Int. J. Solids Structures, 1977, Vol. 13, pp. 293–305. Pergamon Press. Printed in Great Britain

WAVE PROPAGATION IN PERIODIC AND DISORDERED LAYERED COMPOSITE ELASTIC MATERIALS

FRANZ ZIEGLER

Department of Civil Engineering, University of Technology, Vienna, A-1040, Austria

(*Recieved* 29 *March* 1976; *revised* 19 *August* 1976)

Abstract—A powerful complex transfer matrix approach to wave propagation perpendicular to the layering of a composite of periodic and disordered structure is worked out showing propagating and stopping bands of time-harmonic waves and the singular cases of standing waves. A state ratio of left- and right-going plane waves is defined and interpreted geometrically in the complex plane in terms of fixed points and flow lines. For numerical considerations and extension of the approach to higher dimensional problems a continued fraction expansion of the state ratio mapping is presented. Impurity modes of wave propagation in composites with widely spaced impurity cells of different elastic materials are discussed. Stopping bands in the frequency spectrum of global waves in fully disordered composites are found to exist in the range of frequencies corresponding to common gaps in the spectrum of cnstituent regular periodic composites which are constructed from the cells of the disordered system. For those frequencies, waves propagate only a (short) finite distance and are therefore strongly localized modes in a composite of fairly large extent.

1. INTRODUCTION

The gross dynamic behavior of composites has been explored in a large number of papers mostly by approximate theories; for a review see Peck[1]. Only recently has Floquet theory been applied to the analysis of the propagation of Floquet- or Bloch waves through periodically structured composite media. Lee[2] considered a laminated periodically reinforced composite and studied stress-profiles of Floquet waves propagating normal to the layering. Also the band-structure of the frequency spectrum of time-harmonic Floquet waves is recognized in[2–4]. The spectrum shows a strong dispersive character and frequency intervals where no Floquet waves propagate, so called stopping bands. The analysis of the one-dimensional problems in the above mentioned papers makes use of eigenfunctions. Extension to multi-dimensional periodic problems is extremely difficult. The general aspects of waves in periodic structures showing stopping bands and propagating bands is best described in the monograph by Brillouin[5]. Also the texts in solid-state physics, e.g. Ziman[6] and Maradudin *et al.*[7], study the theory of propagating solutions of the Schrödinger equation in periodic lattices and these may be used to describe the mathematically closely analogous elastic wave propagation. The transfer matrix method described in this paper shows this analogy very clearly for both periodic and disordered composites. Disordered lattices are also studied in physics, mainly by employing a Green's function technique. For a continuum this analysis gives a rather formal solution and is extremely difficult to evaluate, see Krumhansl[8]. On the other hand the matrix method employed to the study of disordered composites has also an analog formulation in physics, see e.g. Hori[9]. Recently Mead[10] used a "structural" transfer matrix in connection with a "receptance" function for the description of Floquet waves in periodic structures. The matrix methods are also applicable to two- and three-dimensional wave propagation problems. The formulation shown in this paper makes use of a continued fraction expansion of a complex state ratio of local left- and right-going plane waves, which is extremely advantageous for parameter studies in the engineering applications of composites. In the one-dimensional wave propagation considered here the state ratio is a complex number which changes during propagation thereby running along a circular "flow-line" in the complex plane. The various cases of propagating, nonpropagating and standing Floquet waves are interpreted geometrically. Extension to disordered composites is made and the possibility of stopping bands in the frequency spectrum of global waves is found. The case of a single cell disturbance in a periodic array is treated and a localized so called "impurity mode" at a frequency in the stopping band of the unperturbed periodic composite is described. The result is applicable to composites having impurity cells in a wide distance. The special case of two neighboring impurity cells is also treated in detail showing again the power of the matrix method employed. Transient problems of wave propagation in composites are equally well described by transfer matrix multiplications or the application of the z-transformation to the set of difference equations. The latter is employed in the four pole theory of electric transmission lines which in this difference equation type formulation is also closely related to the study of composites.

2. A LAMINATED COMPOSITE WITH PERIODIC STRUCTURE

A layered composite is considered with density $\rho(x)$ and elastic stiffness $\eta(x)$ varying periodically in x with period a:

$$\rho(x+a)=\rho(x), \qquad \eta(x+a)=\eta(x). \tag{2.1}$$

The length a is the width of a simple cell consisting of one reinforcement sheet of thickness b and one matrix sheet in perfect bond. This double-layer repeats periodically. We assume the various sheets to be homogeneous, the index f referring to reinforcement properties and the index m referring to matrix-sheet parameters.

Plane waves propagating perpendicular to the layering of the alternating linear elastic sheets are solutions of the set of two differential equations:

$$\left.\begin{aligned}\frac{\partial\bar\sigma}{\partial x}&=\rho\frac{\partial^2 U}{\partial t^2}\\ \bar\sigma&=\eta\frac{\partial U}{\partial x}\end{aligned}\right\} \tag{2.2}$$

where $\bar\sigma(x,t)$ denotes stress and U denotes displacement. The equations may describe longitudinal P-waves or transversal SH-waves, respectively. We consider steady oscillatory waves of assigned circular frequency ω, and hence write

$$\bar\sigma(x,t)=\sigma(x)\,\mathrm{e}^{-i\omega t}, \qquad U(x,t)=u(x)\,\mathrm{e}^{-i\omega t} \tag{2.3}$$

to remove the time dependence from the equations. In matrix notation there follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}\begin{Bmatrix}u\\ \sigma\end{Bmatrix}=C\begin{Bmatrix}u\\ \sigma\end{Bmatrix} \tag{2.4}$$

where

$$\mathbf{C}=\begin{Bmatrix}0 & \eta^{-1}\\ -\rho\omega^2 & 0\end{Bmatrix}. \tag{2.5}$$

Solving eqn (2.4) for one homogeneous sheet of arbitrary length 1, displacement and stress at the borders are related by

$$\begin{Bmatrix}u(x+l)\\ \sigma(x+l)\end{Bmatrix}=\mathbf{T}\begin{Bmatrix}u(x)\\ \sigma(x)\end{Bmatrix} \tag{2.6}$$

where

$$\mathbf{T}=\mathrm{e}^{Cl}=\begin{Bmatrix}\cos kl & \dfrac{1}{\eta k}\sin kl\\ -\dfrac{\rho}{k}\omega^2\sin kl & \cos kl\end{Bmatrix}. \tag{2.7}$$

The expansion of the above exponential matrix function may be performed by the Cayley-

Hamilton theorem. The wave number k of plane waves in the homogeneous sheet is defined by $k = \omega\sqrt{(\rho/\eta)}$.

Considering the results (2.6), (2.7) in the reinforcement sheet with parameters $\rho = \rho_f$, $\eta = \eta_f$, $l = b$ and in the matrix sheet with parameters $\rho = \rho_m$, $\eta = \eta_m$, $l = a - b$, respectively, and invoking the continuity of displacement and stress at the common interface of the two sheets, the "structural" transfer of a plane wave of assigned frequency ω through one simple cell of number n is found to be described simply by

$$\begin{Bmatrix} u_{n+1} \\ \sigma_{n+1} \end{Bmatrix} = \mathbf{T}_m \mathbf{T}_f \begin{Bmatrix} u_n \\ \sigma_n \end{Bmatrix} \tag{2.8}$$

where

$$\mathbf{T}_m = \begin{Bmatrix} \cos\beta & \dfrac{1}{\eta_m k_m}\sin\beta \\ -\dfrac{\rho_m}{k_m}\omega^2\sin\beta & \cos\beta \end{Bmatrix}, \qquad \mathbf{T}_f = \begin{Bmatrix} \cos\alpha & \dfrac{1}{\eta_f k_f}\sin\alpha \\ -\dfrac{\rho_f}{k_f}\omega^2\sin\alpha & \cos\alpha \end{Bmatrix} \tag{2.9}$$

and

$$\alpha = k_f b \equiv \omega\tau_f, \quad \beta = k_m(a-b) \equiv \omega\tau_m, \quad k_f = \omega/\sqrt{(\eta_f/\rho_f)}, \quad k_m = \omega/\sqrt{(\eta_m/\rho_m)},$$
$$\tau_f = b/\sqrt{(\eta_f/\rho_f)}, \quad \tau_m = (a-b)/\sqrt{(\eta_m/\rho_m)}. \tag{2.10}$$

For perfect periodic composites the Floquet wave is of the form

$$u(x) = v(x)\,e^{iqx}, \qquad v(x) = v(x+a) \tag{2.11}$$

where $v(x)$ is a periodic function with period a and q denotes the Floquet wave number. From that we have the quasiperiodic boundary condition

$$\begin{Bmatrix} u_{n+1} \\ \sigma_{n+1} \end{Bmatrix} = \begin{Bmatrix} u_n \\ \sigma_n \end{Bmatrix} e^{iqa}. \tag{2.12}$$

With $\lambda = e^{iqa}$, eq (2.8), (2.12) render an eigenvalue problem and the frequency equation

$$\lambda^2 - 2\left[\cos\alpha\cos\beta - \frac{1}{2}\left(p + \frac{1}{p}\right)\sin\alpha\sin\beta\right]\lambda + (\det\mathbf{T}_f)(\det\mathbf{T}_m) = 0 \tag{2.13}$$

where the stiffness ratio

$$p = \eta_f k_f/\eta_m k_m \tag{2.14}$$

is independent of frequency ω. In propagating frequency bands q is real, in stopping bands q is either imaginary or complex. At intermediate frequencies where $q = n\pi/a$, $n = 1,2,3\ldots$ in the extended zone scheme, standing Floquet waves occur. The above formulation renders also the eigenfunctions of displacement and stress and is a straight-forward procedure for periodic composites. For disordered composites difficulties come along because $(\det\mathbf{T}_f)\,(\det\mathbf{T}_m) \neq 1$.

2.1 *A state vector formulation*

A different transfer type formulation considers left and right going waves separately, i.e.

$$u^f = A^f e^{ik_f x} + B^f e^{-ik_f x} \qquad na \le x \le na + b \tag{2.15}$$

$$u^m = A^m e^{ik_m x} + B^m e^{-ik_m x} \qquad na + b \le x \le (n+1)a \tag{2.16}$$

with a similar expression for stress. A two-dimensional state vector may be constructed by considering displacement waves in the form

$$\mathbf{X}_n = \begin{Bmatrix} A_n^f \\ B_n^f \end{Bmatrix} = \begin{Bmatrix} A^f \, e^{ik_f na} \\ B^f \, e^{-ik_f na} \end{Bmatrix}. \tag{2.17}$$

If stresses are also included in the state vector we would end up with a four-dimensional state vector and the state ratio, defined in eqn (2.33), which is a complex number, would become a complex matrix. To avoid this complication, stresses are to be treated separately, if necessary. Requiring continuity of displacement and stress at the interface within the cell, at $x = na + b$, u^m may be expressed by

$$u^m = \left[\frac{1+p}{2} A^f \, e^{i(\alpha + k_f na)} + \frac{1-p}{2} B^f \, e^{-i(\alpha + k_f na)}\right] e^{ik_m(x - na - b)}$$
$$+ \left[\frac{1-p}{2} A^f \, e^{i(\alpha + k_f na)} + \frac{1+p}{2} B^f \, e^{-i(\alpha + k_f na)}\right] e^{-ik_m(x - na - b)} \tag{2.18}$$

Putting

$$u^f[(n+1)a] = A_{n+1}^f + B_{n+1}^f \tag{2.19}$$

and invoking continuity of displacement and stress at the cell interface at $x = (n+1)a$ renders the transfer of displacement through one cell to be

$$\mathbf{X}_{n+1} = \mathbf{Q}\mathbf{X}_n, \tag{2.20}$$

where the complex transfer matrix

$$\mathbf{Q} = \begin{Bmatrix} A & B \\ B^* & A^* \end{Bmatrix} \tag{2.21}$$

is of Cayley type with real determinant

$$\det \mathbf{Q} = AA^* - BB^* = 1. \tag{2.22}$$

The coefficients A and B are given by

$$A = \frac{(1+p)^2}{4p} e^{i(\alpha+\beta)} - \frac{(1-p)^2}{4p} e^{i(\alpha-\beta)} \equiv \frac{1}{1-R^2}[e^{i(\alpha+\beta)} - R^2 e^{i(\alpha-\beta)}]$$
$$B = \frac{p^2-1}{4p}[e^{-i(\alpha+\beta)} - e^{-i(\alpha-\beta)}] \equiv \frac{R}{R^2-1}[e^{-i(\alpha+\beta)} - e^{-i(\alpha-\beta)}] \tag{2.23}$$

where

$$R = \frac{1-p}{1+p} = \frac{\eta_m k_m - \eta_f k_f}{\eta_m k_m + \eta_f k_f}, \tag{2.24}$$

and A^*, B^* are the conjugate complex expression to A, B. From the matrix eigenvalue problem

$$(\mathbf{Q} - \theta\mathbf{I})\mathbf{X} = 0 \tag{2.25}$$

we find the frequency equation,

$$\det(\mathbf{Q} - \theta\mathbf{I}) = 0, \tag{2.26}$$

to be

$$\theta^2 - 2\theta\frac{A + A^*}{2} + 1 = 0 \tag{2.27}$$

with the two roots

$$\theta_{\pm} = (\mathbf{Re}A) \pm \sqrt{((\mathbf{Re}A)^2 - 1)}, \tag{2.28}$$

$$(\mathbf{Re}A) = \frac{1}{1-R^2}[\cos(\alpha+\beta) - R^2\cos(\alpha-\beta)]. \tag{2.29}$$

The eigenvectors $\mathbf{X}_{\pm}$ are solutions of the homogeneous equations

$$(A-\theta_{\pm})X_{\pm}^{(1)} + BX_{\pm}^{(2)} = 0. \tag{2.30}$$

As functions of ω they may be called "limit vectors". In terms of these eigenvectors an arbitrary state vector is expressed by the linear relation

$$\mathbf{X}_n = c_-\mathbf{X}_- + c_+\mathbf{X}_+ \tag{2.31}$$

and after one transfer through a cell

$$\mathbf{X}_{n+1} = \mathbf{Q}\mathbf{X}_n = c_-\theta_-\mathbf{X}_- + c_+\theta_+\mathbf{X}_+ \tag{2.32}$$

when $\mathbf{Q}$ is taken in its diagonal form. The minus sign is given to the eigenvalue with the larger absolut real part. The discussion of three distinct cases corresponding to frequencies in the propagating and stopping bands, respectively and the discrete intermediate frequencies is simplified by the introduction of the complex state ratio

$$z_n = A_n^f/B_n^f \tag{2.33}$$

and its transfer through one cell by the "Möbius-transformation"

$$z_{n+1} = A_{n+1}^f/B_{n+1}^f = (Az_n + B)/(B^*z_n + A^*). \tag{2.34}$$

Equation (2.34) is a one to one conformal mapping of the z_n-plane to a z_{n+1}-plane. Circles in one plane are mapped onto circles of the other plane. The fixed points of the transform are the state ratios of the eigenvectors $X_{\pm}$ and are the roots of the equation

$$z = (Az + B)/(B^*z + A^*) \tag{2.35}$$

which are given in terms of the eigenvalues $\theta_{\pm}$ by

$$z_{\pm} = (B_{\pm} - A^*)/B^* \equiv B/(B_{\pm} - A) = X_{\pm}^{(1)}/X_{\pm}^{(2)}, \qquad z_{\pm}z_{\pm}^* = 1. \tag{2.36}$$

We call $z_{\pm}$ limit points or may call z_- sink point and z_+ source point. Let us consider the three distinct cases $(\mathbf{Re}A)^2 > 1$, $(ReA)^2 < 1$, $(\mathbf{Re}A)^2 = 1$ corresponding to different frequencies ω:

(a) *Stopping bands*: $(\mathbf{Re}A)^2 > 1$. There are finite bands of frequency ω where $(\mathbf{Re}A)^2 > 1$ or

$$\left|\cos[\omega(\tau_f+\tau_m)] - R^2\cos\left[\omega(\tau_f+\tau_m)\frac{\tau_f-\tau_m}{\tau_f+\tau_m}\right]\right| > +(1-R^2), \qquad 0<R<1. \tag{2.37}$$

$\mathbf{Re}A$ is a doubly periodic function of frequency ω and varies within $\pm(1+R^2)/(1-R^2)$, see Fig. 1.

The eigenvalues $\theta_{\pm}$ in this case are real and distinct. After one mapping the state vector $\mathbf{X}_{n+1}$ is obviously nearer to the sink vector $\mathbf{X}_-$, than $\mathbf{X}_n$. Except the special cases $\mathbf{X}_n = c_-\mathbf{X}_-$ or $\mathbf{X}_n = c_+\mathbf{X}_+$, in which case the direction of the state vector is left invariant. Thus, in general, within these frequency intervals, every state vector (except the limit vectors) is "attracted" by the sink vector $\mathbf{X}_-$ and "repulsed" by the source vector $\mathbf{X}_+$. Correspondingly, every point in the complex plane $z_n = A_n^f/B_n^f$, except the limit points $z_{\pm}$, approaches through the Möbius transformation the sink point z_- and recedes from the source point z_+. The transfer matrix is hyperbolic.

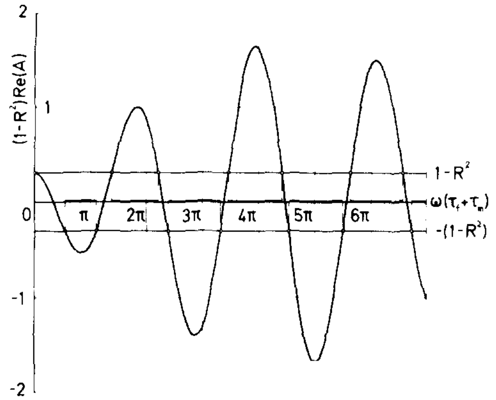

Fig. 1. Propagating and stopping frequency bands. Propagation . . . $(\mathbf{Re}A)^2 < 1$. Spectral gaps . . . $(\mathbf{Re}A)^2 > 1$. For definition of $\mathbf{Re}A$ see eqn (2.29).

$$R = \frac{\sqrt{(\eta_m - \rho_m)} - \sqrt{(\eta_f \rho_f)}}{\sqrt{(\eta_m \rho_m)} + \sqrt{(\eta_f \rho_f)}} = \frac{5}{6} \qquad \frac{\tau_f - \tau_m}{\tau_f + \tau_m} = \frac{\pi}{4}.$$

Geometrically we describe the transformation of the state ratio by a family of circles through the fixed points in the complex plane. Since the state ratio moves along those circles we may them call flow lines.

The orthogonal set of circles correspond to "potential" lines and represent a series of successively mapped circles, see Fig. 2.

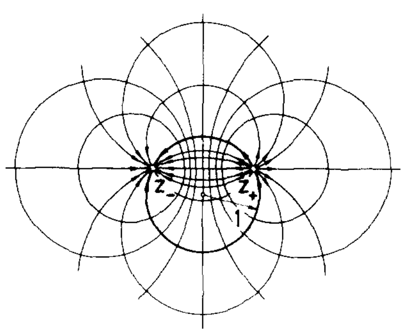

Fig. 2. Flow lines of state ratio to the sink point for stopping frequency ω. State ratio at cell number n: $z_n = A_n^f / B_n^f$, see eqn (2.33)ff. $(\mathbf{Re}A)^2 < 1$, ω in a spectral gap. Sink point . . . z_-. Source point . . . z_+.

The source point is extremely unstable, since any small disturbance in the state ratio corresponding to this point is mapped by one transfer of the wave through a single cell away from the source to the sink.

When the wave propagates through the regular system of cells, repeating periodically, the state ratio is transformed by repeated operation of one and the same linear transformation induced by $\mathbf{Q}$. Without loss of generality we may assume $|z_n| = 1$. After a number of transfers the state ratio has moved on the unit circle close to the sink point and the phase δ_n, where

$$z_n = e^{i\delta_n}, \tag{2.38}$$

changes nearly equal to an integral multiple of 2π, the multiplicity being determined by the nature of the unit cell. Thus, the total amount of phase change over the whole (infinite) system of cells can increase with frequency within this hyperbolic interval at most by a finite amount. This means, there can exist at most only a finite number of eigenvalues in such an interval of ω, which are associated with local modes of propagation. They are determined during the finite number of initial transfers and may be regarded as the "end" effect. In an infinite system they may be neglected. The density of the frequency spectrum must vanish in any of these hyperbolic intervals. They correspond to spectral gaps or stopping bands.

The Floquet wave number, defined by

$$\theta = e^{iqa} \tag{2.39}$$

is imaginary or complex. The phase is, so to speak, locked at the sink phase δ_- and its advance is prevented throughout the hyperbolic interval.

(b) *Propagating bands*: $(\mathbf{Re}A)^2 < 1$. The open intervals of frequency between the stopping bands give distinct conjugate complex eigenvalues θ_-, $\theta_-^* = \theta_+$. Hence, there is no tendency for any point z_n to approach or recede from a fixed point. In fact, the flow lines form a family of circles enclosing the fixed points. The flow lines and the orthogonal set of successively mapped circles through the fixed points are shown in Fig. 3.

The transfer matrix is elliptic. In the perfect periodic system we have propagating waves, the phase δ_n is never locked but can increase freely with frequency ω. Moreover, the total amount of phase change is proportional to the extension of the system. The spectral density does not vanish throughout the elliptic interval, we may speak of propagating bands.

(c)*Standing waves*: $(\mathbf{Re}A)^2 = 1$. In this intermediate case the eigenvalues $\theta_+ = \theta_-$, are real double roots. The fixed points coincide $z_+ = z_-$. The flow lines touch each other at this double point (dipole). Any state ratio z_n approaches this limiting point, see Fig. 4. The transfer matrix is parabolic. The phase is a periodic function over a number of cells. The discrete values of ω correspond to standing waves or purely vibrating cells.

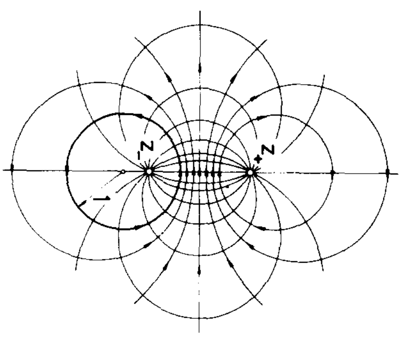

Fig. 3

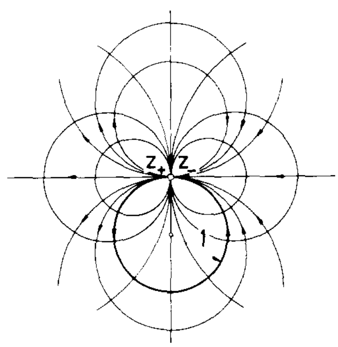

Fig. 4

Fig. 3. Flow lines of state ratio around limiting points for propagating frequency ω. State ratio at cell number n: $z_n = A_n^f/B_n^f$, see eqn (2.33)ff. $(\mathbf{Re}A)^2 > 1$, ω in a propagating band. Limiting points ... z_+, z_-.

Fig. 4. Flow lines of state ratio in case of standing waves. State ratio at cell number n: $z_n = A_n^f/B_n^f$, see eqn (2.33)ff. $(\mathbf{Re}A)^2 = 1$, discrete values of ω corresponding to standing waves. (Coinciding limiting points ... $z_+ = z_-$).

The frequencies in the parabolic case may be calculated from the pair of transzendental equations

$$\tan\omega\frac{\tau_f}{2} = -p\tan\omega\frac{\tau_m}{2},\quad \tan\omega\frac{\tau_f}{2} = -p^{-1}\tan\omega\frac{\tau_m}{2}\quad \text{at}\quad qa = 2k\pi,\qquad (k = 0, 1, 2\ldots) \tag{2.40a}$$

and

$$\tan\omega\frac{\tau_f}{2} = p\cot\omega\frac{\tau_m}{2},\quad \tan\omega\frac{\tau_f}{2} = p^{-1}\cot\omega\frac{\tau_m}{2}\quad \text{at}\quad qa = (2k+1)\pi,\qquad (k = 0, 1, 2\ldots). \tag{2.40b}$$

2.2 *Continued fraction expansion of state ratio mapping*

The same conclusions as from the above geometric interpretation can be drawn from the uniform convergence properties of a continued fraction expansion of the Möbius transformation. For convenience we rewrite the transfer

$$z_n = \frac{A_n^f}{B_n^f} = \frac{-A^* z_{n+1} + B}{B^* z_{n+1} - A} = -\frac{A^*}{B^*}\left(1 - \frac{1}{A^*}\frac{1}{A - B^* z_{n+1}}\right) \tag{2.41}$$

and expand the bracket

$$z_n = \frac{A}{B^*} - \frac{A+A^*}{B^*}[1 - \cfrac{1/(A+A^*)^2}{1 - \cfrac{1/(A+A^*)^2}{1 - \cfrac{1/(A+A^*)^2}{1 - \cfrac{1/(A+A^*)^2}{1 - \cfrac{1/(A+A^*)^2}{1 - \cfrac{1/(A+A^*)^2}{1 - }}}}}} \tag{2.42}$$

The number $w_n = (1/(A+A^*))(A^* + B^* z_n)$ converges uniformly over an interval of frequency ω (a stopping band) when $1/(A+A^*)^2 \leq \frac{1}{4}$ or equivalently $(ReA)^2 \geq 1$ and the value is $|1/(1+w_n) - 4/3| \leq \frac{2}{3}$, due to a theorem by Worpitzky. This formulation is applicable to two- and threedimensional wave propagation problems. The state ratio becomes a matrix and the nominators in the continued fraction are also matrices. For composites of finite extent one may use the above representation or the transfer matrix of a "non-simple" cell, given by $\mathbf{Q}^n$. The latter can be reduced by the Cayley–Hamilton theorem to

$$\mathbf{Q}^n = \left(\theta_+{}^n - \theta_+ \frac{\theta_+{}^n - \theta_-{}^n}{\theta_+ - \theta_-}\right)\mathbf{I} + \frac{\theta_+{}^n - \theta_-{}^n}{\theta_+ - \theta_-}\mathbf{Q}. \tag{2.43}$$

An analogous but more convenient form can be found by substituting

$$\theta_+ = e^{\xi}, \qquad \theta_- = e^{-\xi} \tag{2.44}$$

to be

$$\mathbf{Q}^n = \mathbf{I}\cosh n\xi + \frac{\sinh n\xi}{\sinh \xi}\begin{Bmatrix} \frac{A-A^*}{2} & B \\ B^* & -\frac{A-A^*}{2} \end{Bmatrix}, \qquad \theta_+ \neq \theta_-. \tag{2.45}$$

In the parabolic case, $(\mathbf{Re}A)^2 = 1$:

$$\mathbf{Q}^n = \mathbf{I}\cosh n\xi + n\frac{\cosh n\xi}{\cosh \xi}\begin{Bmatrix} \frac{A-A^*}{2} & B \\ B^* & -\frac{A-A^*}{2} \end{Bmatrix}, \qquad \theta_+ = \theta_- = \mathbf{Re}A. \tag{2.46}$$

The formulation (2.44)–(2.46) is preferred in papers on electric transmission lines and is given by Sauer and Szabó[11, p. 420].

2.3 *Phase and group velocity of propagating Floquet waves*

Substituting $\mathbf{Re}A = \cos(qa)$, where q denotes the wave number of the Floquet wave, the doubly periodic relation

$$(1 - R^2)\cos(qa) = \cos\omega(\tau_f + \tau_m) - R^2 \cos\omega(\tau_f + \tau_m)\frac{\tau_f - \tau_m}{\tau_f + \tau_m} \tag{2.47}$$

may be used to generate an implicit function for the group velocity in propagating bands, $(\mathbf{Re}A)^2 < 1$:

$$c_g = \frac{d\omega}{dq} = \frac{a}{\tau_f + \tau_m}\frac{1 - R^2}{\sin\omega(\tau_f + \tau_m) - \frac{\tau_f - \tau_m}{\tau_f + \tau_m}R^2 \sin\omega(\tau_f + \tau_m)\frac{\tau_f - \tau_m}{\tau_f + \tau_m}}\sin qa, \qquad \omega \neq 0. \tag{2.48}$$

The parabolic case of standing waves in $qa = k\pi$ ($k = 0, 1, 2 \ldots$) renders for $\omega \neq 0$, $c_g = 0$. In the limit $\omega \to 0$, group velocity and phase velocity become equal:

$$\lim_{\substack{\omega\to 0\\ q\to 0}} \frac{d\omega}{dq} = \lim_{\substack{\omega\to 0\\ q\to 0}} \frac{\omega}{q} = \frac{a}{\tau_f + \tau_m}\sqrt{(1 - R^2)} \Big/ \sqrt{\left(1 - R^2\left(\frac{\tau_f - \tau_m}{\tau_f + \tau_m}\right)^2\right)}. \quad (2.49)$$

The high frequency limit of the Floquet phase velocity becomes

$$\lim_{\substack{\omega\to\infty\\ q\to\infty}} \frac{\omega}{q} = a/(\tau_f + \tau_m). \quad (2.50)$$

Frequency spectrum and group velocity are plotted in Fig. 5. The filter properties of a laminated composite can be clearly recognized.

3. A LAMINATED COMPOSITE WITH DISORDERED STRUCTURE

Disorder may be substitutional (cellular) or structural (geometric) in nature. In the course of this paper we assume a geometrically perfect periodic arrangement of cells, which show a slight cellular disorder in the material properties:

$$\rho(x) = \rho_0(x) + \epsilon\rho_1(x) \quad (3.1)$$

$$\eta(x) = \eta_0(x) + \epsilon\eta_1(x) \quad (3.2)$$

where ρ_1 and η_1 are nonperiodic. The mixed system may be described by a set of transfer matrices $\{\mathbf{Q}_i\}$, $i = 0, 1, 2, 3, \ldots$. In regular (periodic) systems the phase is transferred successively by a periodic series of one of the $\mathbf{Q}_i$'s, while in disordered systems it is transferred by a deterministic, or more realistic, by a random series of these matrices.

All quantities pertaining to a particular transfer matrix, e.g. $\mathbf{Q}_j$ are specified by the same index j. For example, the limit vectors and the limit phases are denoted by $\mathbf{X}_\pm^{(j)}$ and $\delta_\pm^{(j)}$, respectively.

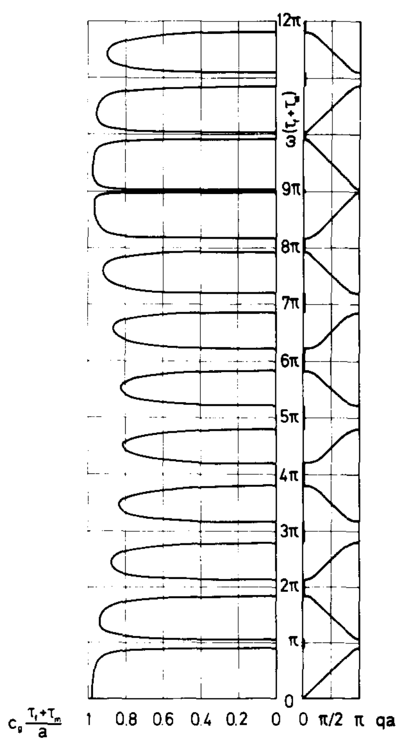

Fig. 5(a).

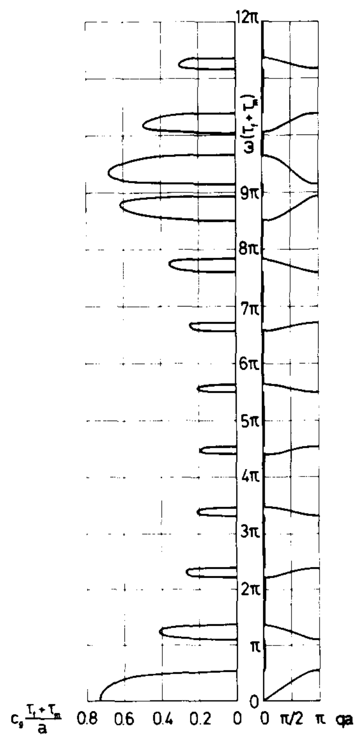

Fig. 5(b).

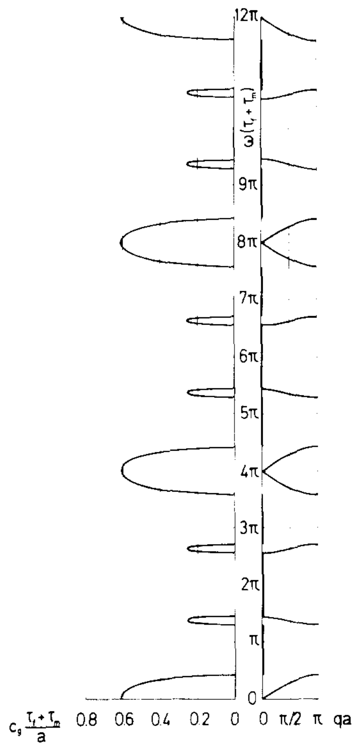

Fig. 5(c).

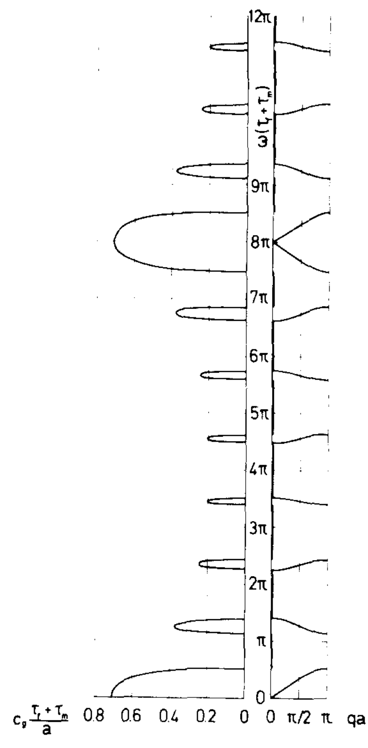

Fig. 5(d).

Fig. 5. Frequency spectrum (reduced zone scheme) and group velocity of Floquet waves. In the stopping bands group velocity is zero.

In the stopping bands group velocity is zero.

(a).
$$R=\frac{\sqrt{(\eta_m\rho_m)}-\sqrt{(\eta_f\rho_f)}}{\sqrt{(\eta_m\rho_m)}+\sqrt{(\eta_f\rho_f)}}=\frac{1}{3},\quad \frac{\tau_f-\tau_m}{\tau_f+\tau_m}=\frac{\pi}{4}$$

Frequency spectrum is nonperiodic.

(b).
$$R=\frac{\sqrt{(\eta_m\rho_m)}-\sqrt{(\eta_f\rho_f)}}{\sqrt{(\eta_m\rho_m)}+\sqrt{(\eta_f\rho_f)}}=\frac{5}{6},\quad \frac{\tau_f-\tau_m}{\tau_f+\tau_m}=\frac{\pi}{4}$$

Frequency spectrum is nonperiodic.

(c).
$$R=\frac{\sqrt{(\eta_m\rho_m)}-\sqrt{(\eta_f\rho_f)}}{\sqrt{(\eta_m\rho_m)}+\sqrt{(\eta_f\rho_f)}}=\frac{5}{6},\quad \frac{\tau_f-\tau_m}{\tau_f+\tau_m}=\frac{1}{2}$$

Frequency spectrum is periodic with period 4π.

(d)
$$R=\frac{\sqrt{(\eta_m\rho_m)}-\sqrt{(\eta_f\rho_f)}}{\sqrt{(\eta_m\rho_m)}+\sqrt{(\eta_f\rho_f)}}=\frac{5}{6},\quad \frac{\tau_f-\tau_m}{\tau_f+\tau_m}=\frac{3}{4}$$

Frequency Spectrum is periodic with period 8π.

Further, the regular periodic system which is described by one of the members of the set, e.g. $\mathbf{Q}_k$, is called the k-th constituent (regular) system of the mixed composite.

Suppose now, that in an interval of ω every transfer matrix belonging to the set $\{\mathbf{Q}_i\}$ is hyperbolic, that is if $(\mathbf{Re}A_i)^2 > \det \mathbf{Q}_i$ for every i, this interval corresponds to a spectral gap for every constituent regular system. In such an interval, all the limit points $z_\pm^{(i)}$ lie on the unit circle. The interval spanned by the set of sink points $\{z_-^{(i)}\}$ will be called sink interval and that spanned by $\{z_+^{(i)}\}$ source interval. Correspondingly, $\{\delta_-^{(i)}\}$ and $\{\delta_+^{(i)}\}$ may be called sink phase—and source phase interval, respectively, see Fig. 6.

Suppose further, that the sink phase and source phase intervals on the unit circle are disjoint. Then, for any ω lying in the common gap of the constituent composites, the phase eventually comes, during successive transformations in the mixed system, into the sink phase interval, and

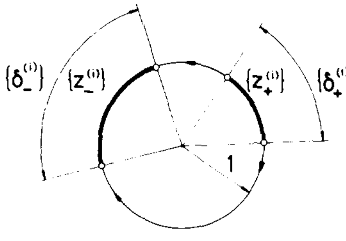

Fig. 6. Flow line of state ratio in case of a frequency ω in the common "stopping band" of the disordered composite. $\{\delta_-^{(1)}\}$ ... source interval. State ratio at cell number N: $z_n = A_n^f/B_n^f$, see eqn (2.33). $(\mathrm{Re}A_i)^2 > \det \mathbf{Q}_i$, $i = 1, 2, 3, \ldots$. For transfer matrix $\mathbf{Q}_i$ see eqn (2.21).

once it has come in, it is trapped in it and can never escape therefrom. Thus, locking of phase occurs.

The results are cast in a theorem due to Hori and Matsuda[9]:

"If (a) an interval of frequency ω corresponds to a stopping band for every constituent regular periodic system, and (b) in that interval the sink- and source phase intervals are disjoint, it gives a spectral gap of the mixed disordered system described by the set $\{\mathbf{Q}_i\}$. The set of matrices $\{\mathbf{Q}_i\}$ may be called a phase disjoint set."

If for a given system the condition (a) is a necessary consequence of the condition (b), we derive the sufficient condition for a forbidden band being the smallest common gap of all constituent regular systems (known in solid state physics as Saxon–Hutner statement).

3.1 *A single cell impurity*

In a regular periodic system described by the transfer matrix $\mathbf{Q}_1$ an impurity pair of laminates forming a cell described by the different transfer matrix $\mathbf{Q}_0$ is embedded. For the value of ω under consideration $\mathbf{Q}_1$ is assumed hyperbolic. The phase at the cell border before the impurity cell is therefore $\delta_-^{(1)}$. Through the impurity cell, there is a possibility that $\delta_-^{(1)}$ is transferred by $\mathbf{Q}_0$ to an appropriate value $\delta_+^{(1)}$, so that the phase reaches any arbitrary value far away from the impurity cell. This descrete value of ω gives an eigenvalue ω_{imp} corresponding to an impurity mode of propagation. Since $\mathbf{Q}_1$ is hyperbolic, ω_{imp} must lie outside the propagating bands of the regular periodic composite. On the other hand, the state vector becomes longer and longer as it approaches the impurity, in fact, it reaches $\mathbf{X}_-^{(1)}$, and shorter and shorter as it leaves it. In other words the eigenmode corresponding to an impurity frequency is strongly localized around the impurity. The amplitude of an impurity mode decays towards either direction like a geometrical series.

In Fig. 7 it is assumed, that the phase actually increases by 2π during one transfer by $\mathbf{Q}_1$, while the phase change by $\mathbf{Q}_0$ is smaller than 2π. For $\omega < \omega_{imp}$ the phase is retarded by the impurity. For $\omega > \omega_{imp}$ a sudden acceleration of phase change occurs, which in physics is called the slip of the phase.

3.2 *Several impurity cells*

The situation depicted in the foregoing paragraph remains essentially the same, if there are many impurities, provided that they are sufficiently apart from one another. Within a narrow

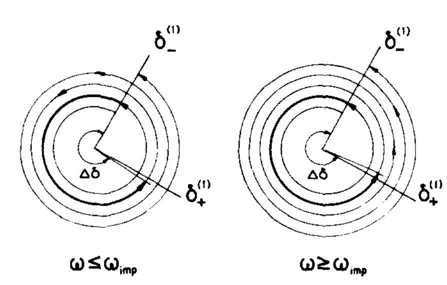

Fig. 7. Change of phase of the state ratio by a single impurity cell for frequency ω in the stopping band of the regular periodic composite. Frequency of localized impurity mode ... ω_{imp}. For $\omega < \omega_{imp}$, the phase of the state ratio is retarded by the single impurity cell. For $\omega > \omega_{imp}$, the phase of the state ratio is accelerated by the single impurity cell. $\Delta\delta$ ... phase change of state ratio in the impurity cell.

interval of ω, many slips occur successively at different impurity cells, and correspondingly there appear a large number of eigenfrequencies around the impurity frequency of an isolated impurity in the stopping band of the regular system.

If two or more inpurity cells are not sufficiently far apart, the phase fails to come into a narrow neighborhood of $\delta_-^{(1)}$ just before the second impurity, so that the situation changes, as depicted in the following Fig. 8.

There are several possibilities for the phase to be transferred from $\delta_-^{(1)}$ through the double impurity cell to $\delta_+^{(1)}$, corresponding to several discrete values of ω in a stopping band of the regular system. These impurity frequencies $\omega_{imp}^{(1)}$, $\omega_{imp}^{(2)}, \ldots$ correspond to localized impurity modes of propagation. In the first case depicted in Fig. 8 the sum of phase changes in both of the impurity cells is less than 2π, corresponding to $\omega_{imp}^{(1)}$, in the second case scetched in Fig. 8 the sum of phase changes overshoots the first one by 2π, creating a second impurity frequency $\omega_{imp}^{(2)}$.

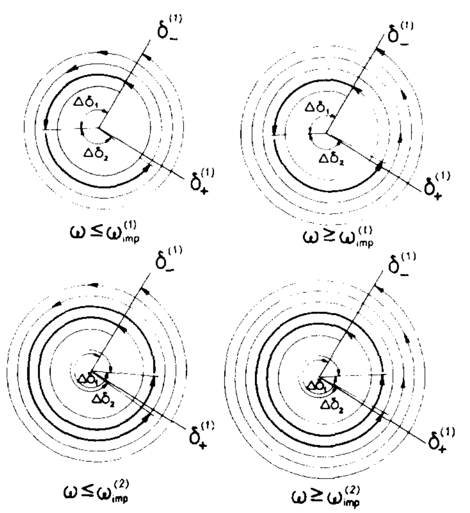

Fig. 8. Change of phase of the state ratio by two neighboring impurity cells for frequency ω in the stopping band of the regular periodic composite. Frequencies of the localized impurity modes ... $\omega_{imp}^{(1)}$, $\omega_{imp}^{(2)}$. In the two cases $\omega < \omega_{imp}^{(1)}$ or $\omega < \omega_{imp}^{(2)}$, the phase of the state ratio is retarded by the impurity cells. In the two cases $\omega > \omega_{imp}^{(1)}$ or $\omega > \omega_{imp}^{(2)}$, the phase of the state ratio is accelerated by the impurity cells. $\Delta\delta_1$ ... phase change of state ratio in the first impurity cell. $\Delta\delta_2$ ... phase change of state ratio in the second impurity cell.

4. CONCLUSIONS

The transfer matrix approach outlined in this paper is suitable for computer studies of various composite materials. It is not restricted to single cells of two layers it may be extended to multi-layered cells repeating periodically or being disordered. In the latter case the stopping bands are easily found from so called sink intervals. Localized impurity modes propagating in the spectral gap of a regular periodic composite surrounding a single or double cell impurity are easily described by the concept of phase transfer, associated with the complex transfer matrix. Extension of the matrix method to include geometric imperfections in the consideration of structural disorder will be later published.

A continued fraction expansion of the state ratio mapping makes it possible to treat two- and threedimensional wave propagation problems in periodic and disordered composites.

Acknowledgements—This work was suggested by Prof. George Herrmann of Stanford University and performed while holding a visiting Professorship in the summer of 1975 in the Division of Applied Mechanics at Stanford University. Partial support of AFOSR Grant 74-2669 and ONR Contract N00014-76-C-0054 as well as a Fulbright Travel Grant are greatfully acknowledged.

REFERENCES

1. J. C. Peck, Stress-wave propagation and fracture in composites. In *Dynamics of Composite Materials* (Edited by E. H. Lee), pp. 8–34. ASME Publ. (1972).
2. E. H. Lee, A survey of variational methods for elastic wave propagation analysis in composites with periodic structures. In *Dynamics of Composite Materials* (Edited by E. H. Lee), pp. 122–138. ASME Publ. (1972).
3. W. Kohn, J. A. Krumhansl and E. H. Lee, Variational Methods for dispersion relations and elastic properties of composite materials. *J. Appl. Mech.* 327–336 (June 1972).

4. E. H. Lee and Wei H. Yang, On waves in composite materials with periodic structure. *SIAM-J. Appl. Math.* **25**, 492–499 (1973).
5. L. Brillouin, *Wave Propagation in Periodic Structures.* Dover, New York (1953).
6. J. M. Ziman, *Principles of the Theory of Solids.* Cambridge Univ. Press (1965).
7. A. A. Maradudin, E. W. Montroll, G. H. Weiss and I. P. Ipatova, Theory of lattice dynamics in the harmonic approximation. *Solid State Physics No.* 3 (Edited by H. Ehrenreich, F. Seitz and D. Turnbull), 2nd Edn. Academic Press, New York (1971).
[8] J. A. Krumhansl, Randomness and wave propagation in inhomogeneous media. In *Dynamics of Composite Materials* (Edited by E. H. Lee), 139–150 ASME Publ. (1972).
9. J. Hori, Spectral properties of disordered chains and lattices. *Int. Ser. of Monographs in Natural Philosophy* (Edited by D. Ter. Haar), Vol. 16. Pergamon Press, Oxford (1968).
10. D. J. Mead, Wave propagation and natural modes in periodic systems—I and II. *J. Sound and Vibration* **40**, 1–39 (1975).
11. R. Sauer and I. Szabó (Editors), *Mathematische Hilfsmittel des Ingenieurs.* Part I by G. Doetsch, F. W. Schäfke and H. Tietz. *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen* Vol. 139, (Chap. VI). Springer-Verlag, Berlin (1967).